LECTURE NOTES - ADVANCED TOPICS IN MATHEMATICAL LOGIC

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Abstract. Lecture notes from the summer 2016 in Bonn by Philipp Lücke and Philipp Schlicht. We study forcing axioms and their applications. The topics include supercompact cardinals, the proper forcing axiom, the forcing axiom for Axiom A forcings of size continuum, the tree property for $\aleph_2$.

Contents

1. The proper forcing axiom

We give proofs of the consistency of the proper forcing axiom PFA from a supercompact cardinal and the consistency of the forcing axiom for Axiom A forcing of size continuum from a weakly compact cardinal. April 12

1.1. **Supercompact cardinals.** The iterated forcings below use a supercompact cardinal. Supercompact cardinals (and large cardinals in general) state that the universe is tall in a well-defined sense.

Definition 1.1.1. Suppose that F is a filter on a set S and κ is a cardinal.

(a) F is *$<\kappa$-complete* if for all $\langle X_i \mid i < \alpha\rangle$ with $\alpha < \kappa$ and $X_i \in F$ for all $i < \alpha$, $\bigcap_{i<\alpha} X_i \in F$.
(b) F is *principal* if it $\{i\} \in F$ for some $i \in S$.
(c) κ is *measurable* if there is a non-principal $<\kappa$-complete ultrafilter on κ.

Supercompact cardinals can be defined by filters on $P_\kappa(\lambda)$, where κ, λ are cardinals with $\kappa \leq \lambda$.

Definition 1.1.2. Suppose that κ, λ are cardinals with $\kappa \leq \lambda$.

(a) $P_\kappa(\lambda) = \{A \subseteq \lambda \mid |A| < \kappa\}$.
(b) $\hat{x} = \{y \in P_\kappa(\lambda) \mid x \subseteq y\} \in U$ for $x \in P_\kappa(\lambda)$.
(c) A filter on $P_\kappa(\lambda)$ is *uniform* if $\hat{x} \in U$ for all $x \in P_\kappa(\lambda)$.
(d) A filter on $P_\kappa(\lambda)$ is *fine* if it is $<\kappa$-complete and uniform.

An example for a filter on $P_\kappa(\lambda)$ is the *club filter*.

Example 1.1.3. Suppose that κ, λ are cardinals with $\kappa \leq \lambda$. Suppose that C is a subset of $P_\kappa(\lambda)$.

(a) C is *unbounded* if for every $x \in P_\kappa(\lambda)$, there is some $y \in C$ with $x \subseteq y$.

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(b) C is *closed* if for every $\subseteq$-increasing chain $\langle x_\alpha \mid \alpha < \gamma\rangle$ with $\gamma < \kappa$ and $x_\alpha \in C$ for all $\alpha < \gamma$, $\bigcup_{\alpha<\gamma} x_\alpha \in C$.
(c) C is *club* if it is closed and unbounded.

The *club filter* $\mathrm{Club}_{P_\kappa(\lambda)}$ on $P_\kappa(\lambda)$ is defined as the set of subsets D of $P_\kappa(\lambda)$ such that there is a club C in $P_\kappa(\lambda)$ with $C \subseteq D$.

Definition 1.1.4. Suppose that κ, λ are cardinals with $\kappa \leq \lambda$.

(a) Suppose that $\vec{X} = \langle X_i \mid i < \lambda\rangle$ is a sequence of subsets of $P_\kappa(\lambda)$. The *diagonal intersection* of $\vec{X}$ is defined as
$$\triangle\vec{X} = \triangle_{i<\lambda} X_i = \{x \in P_\kappa(\lambda) \mid x \in \bigcap_{i\in x} X_i\}.$$
(b) Suppose that $\vec{X} = \langle X_a \mid a \in P_\omega(\lambda)$ is a sequence of subsets of $P_\kappa(\lambda)$. The *diagonal intersection* of $\vec{X}$ is defined as
$$\triangle\vec{X} = \triangle_{a\in P_\omega(\lambda)} X_a = \{x \in P_\kappa(\lambda) \mid x \in \bigcap_{a\in P_\omega(\lambda),\ a\subseteq x} X_a\}.$$
(c) Suppose that $X \subseteq P_\kappa(\lambda)$. A function $f\colon X \to \lambda$ is *regressive* if $f(x) \in x$ for all $x \in X$.
(d) Suppose that $X \subseteq P_\kappa(\lambda)$. A function $f\colon X \to P_\omega(\lambda)$ is *regressive* if $f(x) \subseteq x$ for all $x \in X$.

The following is an analogue to Fodor's lemma.

Definition 1.1.5. Suppose that κ, λ are cardinals with $\kappa \leq \lambda$.

(a) Suppose that F is a filter on $P_\kappa(\lambda)$. The set F^+ of *F-positive sets* is defined as
$$F^+ = \{x \in P_\kappa(\lambda) \mid \forall y \in F\ x \cap y \neq \emptyset\}.$$
(b) An filter F on $P_\kappa(\lambda)$ is *normal* if it is fine and the following condition holds. Suppose that $X \in F^+$ and $f\colon X \to \lambda$ is regressive. Then there is a set $Y \subseteq X$ in F^+ such that $f \upharpoonright Y$ is constant.

Example 1.1.6. Suppose that κ, λ are cardinals with $\kappa \leq \lambda$. Then $\mathrm{Club}^+_{P_\kappa(\lambda)}$ is the set of stationary subsets of $P_\kappa(\lambda)$.

Lemma 1.1.7. *Suppose that κ, λ are cardinals with $\kappa \leq \lambda$. Suppose that F is a $<\kappa$-complete filter on $P_\kappa(\lambda)$. Suppose that $\gamma < \kappa$ and $\langle X_i \mid i < \gamma\rangle$ is a sequence with $X_i \notin F^+$ for all $i < \gamma$. Let $X = \bigcup_{i<\gamma} X_i$. Then $X \notin F^+$.*

Proof. There is a set $C_i \in F$ with $C_i \cap X_i = \emptyset$ for every $i < \gamma$. Let $C = \bigcap_{i<\gamma} C_i \in F$. Then $C \cap X = \emptyset$. □

Lemma 1.1.8. *Suppose that κ, λ are cardinals with $\kappa \leq \lambda$. Suppose that $F$is a filter on $P_\kappa(\lambda)$. The following conditions are equivalent.*

(1) *F is normal.*
(2) *For every sequence $\vec{X} = \langle X_i \mid i < \lambda\rangle$ with $X_i \in F$ for all $i < \lambda$, $\triangle\vec{X} \in F$.*
(3) *If $X \in F^+$ and $f\colon X \to P_\omega(\lambda)$ is regressive, then there is a set $Y \subseteq X$ in F^+ such that $f \upharpoonright Y$ is constant.*
(4) *For every sequence $\vec{X} = \langle X_a \mid a \in P_\omega(\lambda)\rangle$ with $X_a \in F$ for all $a \in P_\omega(\lambda)$, $\triangle\vec{X} \in F$.*

Proof. Suppose that (1) holds. To prove (2), suppose that $\vec{X} = \langle X_i \mid i < \lambda\rangle$ and $X_i \in F$ for all $i < \lambda$. Suppose that $\triangle\vec{X} \notin F$. Then $P_\kappa(\lambda) \setminus \triangle\vec{X} \in F^+$. Let $f\colon P_\kappa(\lambda) \setminus \triangle\vec{X} \to \lambda$, where $f(x)$ is defined as the least $i \in x$ such that $x \notin C_i$. There is some $Y \in F^+$ such that $f \upharpoonright Y$ is constant with value $i < \lambda$ by the assumption. Then $Y \cap C_i = \emptyset$. This contradicts the fact that $Y \in F^+$.

Suppose that (2) holds. To prove (1), suppose that $X \in F^+$ and $f\colon X \to \lambda$ is regressive. Suppose that the conclusion of (1) fails. Then for every $i \in X$, there is some set $C_i \in F$ such that $f(x) \neq i$ for all $x \in C_i$. Let $C_i = X$ for $i \notin X$. Let $C = \triangle_{i<\lambda} C_i \in F$. Suppose that $x \in C$. Then $f(x) \in C_i$ for all $i \in x$, hence $f(x) \neq i$. This contradicts the assumption that f is regressive.

The equivalence of (3) and (4) is analogous.

Suppose that (1) holds. To prove (3), suppose that $X \in F^+$ and $f\colon X \to P_\omega(\lambda)$ is regressive. Then there is a set $Y \subseteq X$ in F^+ such that $f \upharpoonright Y$ is constant. Let $X_n = \{y \in Y \mid |f(y)| = n\}$ for $n \in \omega$. There is some $n \in \omega$ with $X_n \in F^+$ by Lemma 1.1.7. We prove the claim by induction on n. Let $g\colon X_n \to \lambda$, $g(x) = \min(f(x))$. There is a subset $Y \in F^+$ of X_n such that $g \upharpoonright Y$ is constant by (1). Let $h\colon Y \to \lambda$, $h(x) = f(x) \setminus \{\min(x)\}$. There is some subset $\bar{Y} \in F^+$ of Y such that $h \upharpoonright \bar{Y}$ is constant. Hence $f \upharpoonright \bar{Y}$ is constant.

Moreover (3) implies (1). $\square$

Definition 1.1.9. Suppose that U is an ultrafilter on a set S.

(a) $f \sim_U g$ if $\{x \in S \mid f(x) = g(x)\} \in U$ for $f, g\colon S \to X$.
(b) $[f] = [f]_U = \{g\colon S \to V,\ g$ has minimal rank with $f \sim_U g\}$.
(c) $\mathrm{Ult}(V,U) = \{[f] \mid f\colon S \to V\}$.
(d) $[f] \in_U [g]$ if $\{x \in S \mid f(x) \in g(x)\} \in U$ for $f, g\colon S \to X$.

We write id for the identity function on S.

Lemma 1.1.10 (Los)**.** *Suppose that U is an ultrafilter on a set S.*

(1) *For every formula $\varphi(x_0, \ldots, x_n)$ and $f_0, \ldots, f_n\colon S \to X$*
$$\mathrm{Ult}(V,U) \vDash \varphi([f_0], \ldots, [f_n]) \Leftrightarrow \{x \in S \mid \varphi(f(\alpha_0), \ldots, f(\alpha_n))\} \in U.$$
(2) *$j_U\colon V \to \mathrm{Ult}(V,U)$, $j_U(x) = [c_x]$, $c_x(i) = x$ for all $i \in S$, is an elementary embedding.*

Proof. (1) This is proved by induction on the complexity of formulas (see [Theorem 12.3, Jech]).

(2) This follows from (1). $\square$

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Lemma 1.1.11. *Suppose that U is an $< \omega_1$-complete ultrafilter on a set S. Then $\mathrm{Ult}(V,U)$ is well-founded.*

Proof. Suppose that $\langle f_n \mid n \in \omega\rangle$ is a sequence of functions $f_n\colon S \to V$ with $[f_{n+1}] \in_U [f_n]$ for all $n \in \omega$. Then $S_n = \{s \in S \mid f_{n+1}(s) \in f_n(s)\} \in U$ for all $n \in \omega$. Since U is $< \omega_1$-complete, $\bar{S} = \bigcap_{n\in\omega} S_n \in U$. Let $s \in \bar{S}$. Then $\langle f_n(s) \mid n \in \omega\rangle$ is strictly $\in$-decreasing, contradicting the well-foundedness of $\in$. $\square$

If U is an $< \omega_1$-complete ultrafilter on a set S, we will identify the ultrapower $\mathrm{Ult}(V,U)$ with its transitive collapse.

Lemma 1.1.12. *Suppose that U is an $< \omega_1$-complete ultrafilter on a set S. Then $\mathrm{Ord}^{\mathrm{Ult}(V,U)} = \mathrm{Ord}$.*

Proof. The definition of the class Ord of ordinals is Δ_0 and hence absolute between transitive classes. Hence $\mathrm{Ord}^{\mathrm{Ult}(V,U)} \subseteq \mathrm{Ord}$.

Claim 1.1.13. $\mathrm{Ord}^{\mathrm{Ult}(V,U)}$ is transitive.

Proof. Suppose that $x \in y \in \mathrm{Ord}^{\mathrm{Ult}(V,U)}$. Since $\mathrm{Ult}(V,U)$ is transitive, $x \in \mathrm{Ult}(V,U)$. Since $\mathrm{Ult}(V,U) \vDash \mathrm{Ord}^{\mathrm{Ult}(V,U)}$ is transitive, $x \in \mathrm{Ord}^{\mathrm{Ult}(V,U)}$. $\square$

Since $j_U[\mathrm{Ord}] \subseteq \mathrm{Ord}^{\mathrm{Ult}(V,U)}$, $\mathrm{Ord}^{\mathrm{Ult}(V,U)}$ is a proper class. Hence $\mathrm{Ord}^{\mathrm{Ult}(V,U)} = \mathrm{Ord}$. $\square$

Definition 1.1.14. Suppose that $j\colon V \to M$ is an elementary embedding into a transitive class. Let $\mathrm{crit}(j)$ denote the least ordinal α with $j(\alpha) \neq \alpha$.

Lemma 1.1.15. *Suppose that U is a $<\kappa$-complete ultrafilter on a set S. Then* $\mathrm{crit}(j_U) \geq \kappa$.

Proof. We show that $[c_\gamma] = \gamma$ for all $\gamma < \kappa$. Suppose that $\gamma < \kappa$ and $[c_\alpha] = \alpha$ for all $\alpha < \gamma$. Suppose that $\gamma < \kappa$ and $[f] \in [c_\gamma]$. Then $f(i) \in \gamma$ on a set S in U. Let $S_\alpha = \{i \in S \mid f(i) = \alpha\}$ for $\alpha < \gamma$. Since U is $<\kappa$-complete, $S_\alpha \in U$ for some $\alpha < \gamma$. Then $[f] = [c_\alpha] = \alpha$. Hence $[c_\gamma] = \gamma$. $\square$

Lemma 1.1.16. *Suppose that U is an $<\omega_1$-complete ultrafilter on a set S, X, Y are sets and α is an ordinal.*

(1) *If $j[X] \in \mathrm{Ult}(V,U)$, $Y \subseteq \mathrm{Ult}(V,U)$ and $|Y| \leq |X|$, then $Y \in \mathrm{Ult}(V,U)$.*
(2) *$j[\alpha] \in \mathrm{Ult}(V,U)$ if and only if $\mathrm{Ult}(V,U)^\alpha \subseteq \mathrm{Ult}(V,U)$.*

Proof. (1) Suppose that $Y = \{[f_x] \mid x \in X\}$. There is a function $g\colon S \to P(X)$ with $[g] = j[X]$ by Lemma 1.1.10. Let $h\colon S \to V$ such that $h(i)$ is a function with domain $g(i)$ and for all $x \in g(i)$, $h(i)(x) = f_x(i)$.

Then $\mathrm{dom}([h]) = [g] = j[X]$ by Lemma 1.1.10. Then $[h](j(x)) = [f_x]$ for all $x \in X$ by Lemma 1.1.10, since $\{i \in S \mid h(i)(c_x(i)) = f_x(i)\} = S \in U$. Hence $\mathrm{ran}([h]) = j[X]$.

(2) This follows from (1). $\square$

Lemma 1.1.17. *Suppose that κ, λ are cardinals with $\kappa \leq \lambda$. Suppose that U is a normal ultrafilter on $P_\kappa(\lambda)$.*

(1) *$\mathrm{crit}(j) = \kappa$.*
(2) *If U is normal, then $\mathrm{Ult}(V,U)^\lambda \subseteq \mathrm{Ult}(V,U)$.*

Proof. (1) $\mathrm{crit}(j) \geq \kappa$ by Lemma 1.1.15. Let $f\colon P_\kappa(\lambda) \to \kappa$, $f(x) = \mathrm{otp}(x)$. For every $\alpha < \kappa$, $[c_\alpha] < [f]$, since $\{x \in P_\kappa(\lambda) \mid \alpha < \mathrm{otp}(x)\} \supseteq \widehat{(\alpha+1)} \in U$. Hence $\mathrm{crit}(j) = \kappa$.

(2) It is sufficient to show that for every subset Y of $\mathrm{Ult}(V,U)$ of size λ, $Y \in \mathrm{Ult}(V,U)$. Suppose that $\langle a_\alpha \mid \alpha < \lambda\rangle$ is a sequence with $a_\alpha = [f_\alpha]$ for $\alpha < \kappa$. We define $f\colon P_\kappa(\lambda) \to V$, $f(x) = \{f_\alpha(x) \mid \alpha \in x\}$.

Claim. $[f] = \{a_\alpha \mid \alpha < \lambda\}$.

Proof. Suppose that $\alpha < \lambda$. Since U is fine, $\widehat{\{\alpha\}} = \{x \in P_\kappa(\lambda) \mid \alpha \in x\} \in U$. Hence $[f_\alpha] \in [f]$.

Suppose that $[g] \in [f]$. Since U is normal, there is some $\alpha \in x$ such that for almost all $x \in P_\kappa(\lambda)$ (i.e. on a set in U), $g(x) = f_\alpha(x)$. Then $[g] = [f_\alpha] = a_\alpha$. $\square$

This completes the proof. $\square$

Lemma 1.1.18. *Suppose that κ, λ are cardinals with $\kappa \leq \lambda$. Suppose that U is a normal ultrafilter on $P_\kappa(\lambda)$.*

(1) *For all $X \in P_\kappa(\lambda)$, $X \in U$ if and only if* $[\mathrm{id}] \in j(X)$.
(2) $[\mathrm{id}] = j[\lambda]$.

Proof. (1) $[\mathrm{id}] \in j(X)$ holds if and only if $\{x \in P_\kappa(\lambda) \mid x \in c_X(x)\} = X \in U$ by Lemma 1.1.10.

(2) Suppose that $\alpha \in j[\lambda]$. Suppose that $\gamma < \lambda$ and $j(\gamma) = \alpha$. Since U is fine, $\check{\{\gamma\}} = \{x \in P_\kappa(\lambda) \mid \gamma \in x\} \in U$. Hence $j(\gamma) = [c_\gamma] \in [\mathrm{id}]$ by Lemma 1.1.10.

Suppose that $[f] \in [\mathrm{id}]$. Then $f(x) \in x$ on a set S in U. Since U is normal, there is a subset $T \in U$ of S and some y such that $f(x) = y$ for all $x \in T$. Then $y \in P_\kappa(\lambda)$ and $[f] = [c_y] = j(y)$ by Lemma 1.1.10. $\square$

Definition 1.1.19. Suppose that κ, λ are cardinals with $\kappa \leq \lambda$.

(i) An elementary embedding $j\colon V \to M$ is called λ-*supercompact* if M transitive, $M^\lambda \subseteq M$, and $j(\kappa) > \lambda$ for $\kappa = \mathrm{crit}(j)$.
(ii) A cardinal κ is λ-*supercompact* for some cardinal $\lambda \geq \kappa$ if and only if there is a λ-supercompact embedding j with $\kappa = \mathrm{crit}(j)$.
(iii) A cardinal κ is *supercompact* if it is λ-supercompact for all $\lambda \geq \kappa$.

Supercompactness is very high in the large cardinal hierarchy. For example, every supercompact cardinal is measurable and there are many measurable cardinals below it.

Lemma 1.1.20. *Suppose that κ, λ are uncountable cardinals with $\kappa \leq \lambda$. The following conditions are equivalent.*
(a) *κ is λ-supercompact.*
(b) *There is an elementary embedding $j\colon V \to M$ into some transitive class M with* $\mathrm{crit}(j) = \kappa$, *$j(\kappa) > \lambda$ and $j[\lambda] \in M$.*
(c) *There is a normal ultrafilter on $P_\kappa(\lambda)$.*

Proof. The implication from (a) to (b) follows from the definition of λ-supercompact embeddings.

Suppose that (b) holds. Suppose that j is λ-supercompact with $\mathrm{crit}(j) = \kappa$. Let
$$U = U_j = \{X \subseteq P_\kappa(\lambda) \mid j[\lambda] \in j(X)\}.$$

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Claim 1.1.21. U is a ultrafilter.

Proof. $P_\kappa(\lambda)) \in U$, since $j(\kappa) > \lambda$ and $j[\lambda] \in P_{j(\kappa)}(j(\lambda))^M = j(P_\kappa(\lambda))$. The remaining properties of ultrafilters follow from the definition of U and from the assumption that j is elementary. $\Box$

Claim 1.1.22. U is non-principal.

Proof. Suppose that $x \in P_\kappa(\lambda)$ and $\{x\} \in U$. Then $j[\lambda] \in j(\{x\}) = \{j(x)\}$ and hence $j[\lambda] = j(x)$. Then $j(\mathrm{otp}(x)) = \mathrm{otp}(j(x)) = \mathrm{otp}(j[\lambda]) = \lambda$. This contradicts the assumption that $j(\kappa) > \lambda$. $\Box$

Claim 1.1.23. U is $< \kappa$-complete.

Proof. Suppose that $\vec{X} = \langle X_i \mid i < \gamma\rangle$ of sets $X_i \in U$ of length $\gamma < \kappa$. Let $X = \bigcap_{i<\gamma} X_i$. Since $j(\gamma) = \gamma$, we have $j(\vec{X}) = \langle j(X_i) \mid i < \gamma\rangle$ and $j(X) = \bigcap_{i<\gamma} j(X_i)$. Hence $j[\lambda] \in j(X)$. $\Box$

Claim 1.1.24. U is fine.

Proof. Suppose that $x \in P_\kappa(\lambda)$. Then $j(x) = j[x]$. Suppose that $\langle x_\alpha \mid \alpha < \gamma\rangle$ enumerates x. Since $j(\gamma) = \gamma$, $j(\langle x_\alpha \mid \alpha < \gamma\rangle) = \langle j(x_\alpha) \mid \alpha < \gamma\rangle$ and $j(x) = j[x] \subseteq j[\lambda]$. Hence $j[\lambda] \in \hat{j(x)} = j(\hat{x})$. Hence $\hat{x} \in U$. $\Box$

Claim 1.1.25. U is normal.

Proof. Suppose that $\vec{X} = \langle X_i \mid i < \lambda\rangle$ is a sequence of elements of U. We claim that $j[\lambda] \in j(\triangle\vec{X}) = \triangle j(\vec{X})$. Suppose that $\gamma \in j[\lambda]$. Then there is some $\alpha < \lambda$ with $j(\alpha) = \gamma$. Since $X_\alpha \in U$, $j[\lambda] \in j(X_\alpha) = j(\vec{X})_{j(\alpha)} = j(\vec{X})_\gamma$. $\Box$

Suppose that (c) holds. Suppose that U is a normal ultrafilter on $P_\kappa(\lambda)$.

Claim 1.1.26. $j_U(\kappa) > \lambda$.

Proof. Let $f\colon P_\kappa(\lambda) \to \kappa$, $f(x) = \mathrm{otp}(x)$. Since $[\mathrm{id}] = j[\lambda]$ and $\mathrm{otp}(j[\lambda]) = \lambda$, $[f] = \lambda$ by Los' theorem. Moreover $[f] \in [c_\kappa]$ by Los' theorem. $\Box$

This completes the proof. □

Lemma 1.1.27. *Suppose that κ is an uncountable cardinal. The following conditions are equivalent.*

(a) *κ is measurable.*
(b) *κ is κ-supercompact.*
(c) *There is an elementary embedding $j\colon V \to M$ into a transitive class M with $\mathrm{crit}(j) = \kappa$.*

Proof. Suppose that (a) holds. Suppose that U is a non-principal $<\kappa$-complete ultrafilter on κ. Then $j_U[\kappa] = \kappa$. Then j_U satisfies (b) by Lemma 1.1.15 and Lemma 1.1.16.

The implication from (b) to (c) follows from Lemma 1.1.20.

Suppose that (c) holds. Then $j[\kappa] = \kappa$. Hence (a) follows from Lemma 1.1.20. □

Theorem 1.1.28. *An uncountable cardinal κ is supercompact if and only if for every $\eta > \kappa$, there is an $\alpha < \kappa$ and $i\colon V_\alpha \to V_\eta$ with $i(\mathrm{crit}(i)) = \kappa$.*

Proof. Suppose that $j\colon V \to M$ is $|V_\eta|$-supercompact with $\mathrm{crit}(j) = \kappa$. Then $V_\alpha^M = V_\alpha$ for all $\alpha \leq \eta$ by induction on α. Then $j \restriction V_\eta\colon V_\eta \to V^M_{j(\eta)}$ is a element of M.

In M, there is some $\bar\eta$ and an elementary embedding $i\colon V_{\bar\eta} \to V^M_{j(\eta)}$ with $i(\mathrm{crit}(i)) = j(\kappa)$. Since j is elementary, in V there is some $\bar\eta$ and an elementary embedding $i\colon V_{\bar\eta} \to V_\eta$ with $i(\mathrm{crit}(i)) = \kappa$.

For the converse, suppose that $\gamma \geq \kappa$ and $\delta = \gamma + \omega$. Suppose that $\beta < \kappa$ and $i\colon V_\beta \to V_\delta$ with $i(\mathrm{crit}(i)) = \kappa$. Then $\beta = \alpha + \omega$ for some $\alpha < \kappa$. Then $i[\alpha] \in P_\kappa(\gamma)$. We define an ultrafilter U on $P_{\mathrm{crit}(i)}(\alpha)$ by

$$X \in U \Leftrightarrow i[\alpha] \in i(X).$$

As in the proof of Lemma 1.1.20, U is a normal ultrafilter on $P_{\mathrm{crit}(i)}(\alpha)$ in V_β. Since i is elementary, there is a normal ultrafilter on $P_\kappa(\gamma)$ in V_δ and therefore in V. □

1.2. **Some Lemmas on forcing and names.** We begin with preliminary results on forcing names and on iterated forcing. Let $\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{S}$ always denote partial orders and $\dot{\mathbb{P}}, \dot{\mathbb{Q}}, \dot{\mathbb{R}}, \dot{\mathbb{S}}$ names for partial orders. Recall that $H_\kappa = \{x \mid |tc(x)| < \kappa\}$, where κ is a cardinal.

Lemma 1.2.1. *If $\mathbb{P}$ is a forcing that does not collapse κ and $\dot{x} \in H_\kappa$, then $p \Vdash \dot{x} \in H_\kappa$ for any $p \in \mathbb{P}$.*

Proof. By induction on $\mathrm{rk}(\dot{x})$. The lemma holds for $\mathrm{rk}(\dot{x}) = 0$, so suppose that it is true for all names with rank smaller $r = \mathrm{rk}(\dot{x})$. Suppose that $\dot{x} \in H_\kappa$ and write $\dot{x} = \{(\dot{y}_i, p_i) \mid i \in I\}$ for some indexing set I with $|I| = \kappa$. By the induction hypothesis, $\mathbb{1}_{\mathbb{P}} \Vdash \dot{y}_i \in H_\kappa$. Since $|\dot{x}| < \kappa$ and κ remains a cardinal, $\mathbb{1}_{\mathbb{P}} \Vdash |\dot{x}| < \kappa$. Thus $\mathbb{1}_{\mathbb{P}} \Vdash \dot{x} \in H_\kappa$. □

The reversal of this result is more interesting.

Lemma 1.2.2. *(Goldstern) If κ is regular and $\mathbb{P} \subseteq H_\kappa$ and satisfies the κ-c.c., then for all $p \in \mathbb{P}$: If $p \Vdash \sigma \in H_\kappa$, there is $\dot\sigma \in H_\kappa$ with $p \Vdash \sigma = \dot\sigma$.*

Proof.

Claim. *For every $x \in H_\kappa$, there is some $\lambda < \kappa$ and a sequence $(x_\alpha \mid \alpha \leq \lambda)$, $x_\alpha \in H_\kappa$ such that: for all $\alpha \leq \lambda : x_\alpha \subseteq \{x_\beta \mid \beta < \alpha\}$ and $x = x_\lambda$.*

Proof. We prove this via induction on x, it is clear for $x = \emptyset$. Suppose that this holds for all $y \in x$ and take for each $y \in x$ an appropriate $\lambda^y < \kappa$ and one such sequence $(x^y_\alpha \mid \alpha \leq \lambda^y)$. Let $\lambda = \sup_{y \in x} \lambda^y$. $\lambda < \kappa$, since $|x| < \kappa$ and κ is regular. Let $(x_\alpha)_{\alpha < \lambda}$ be the concatenation of all the $(x^y_\alpha)_{\alpha \leq \lambda^y}$ and finally set $x_\lambda = x$. This works because every $y \in x$ is at some point in the sequence. □

Since $\mathbb{P}$ satisfies the κ-c.c., it does not collapse κ. Now suppose that $p \in \mathbb{P}$ and $p \Vdash \sigma \in H_\kappa$. Then we can find names $\dot{\lambda}$, $\dot{x}_\alpha$ for the sequence discussed above. There is an ordinal $\lambda < \kappa$ such that $p \Vdash \dot{\lambda} \leq \check{\lambda}$ and since, in $V[G]$, we may set $x_\alpha = \emptyset$ for all $\dot{\lambda}^G < \alpha < \check{\lambda}^G$, we can assume that $\dot{\lambda} = \check{\lambda}$.

We now inductively define $\dot{\sigma}_\alpha$. For every $\beta < \alpha$, we choose an antichain $A_p^{\alpha,\beta}$ consisting of conditions $q \leq p$ with $q \Vdash_{\mathbb{P}} \sigma_\beta \in \sigma_\alpha\}$ and such that $A_p^{\alpha,\beta}$ is maximal with this property. Let $\dot{\sigma}_\alpha := \{(\dot{\sigma}_\beta, q) \mid \beta < \alpha \wedge q \in A_p^{\alpha,\beta}\}$. Let $\dot{\sigma} = \dot{\sigma}_\lambda$. Then by induction, all $\dot{\sigma}_\alpha$ are in H_κ.

We now show that for all $\alpha < \lambda$, $p \Vdash \sigma_\alpha = \dot{\sigma}_\alpha$, in particular $p \Vdash \sigma = \dot{\sigma}$. To prove this by induction, suppose that for all $\beta < \alpha$, $p \Vdash \sigma_\beta = \dot{\sigma}_\beta$. Suppose that G is $\mathbb{P}$-generic with $p \in G$. Then

$$\begin{aligned} \dot{\sigma}_\alpha^G &= \left\{\dot{\sigma}_\beta^G \mid \beta < \alpha \wedge \exists q \leq p : q \in G \wedge q \Vdash \sigma_\beta \in \sigma_\alpha\right\} \text{ (by definition)} \\ &= \left\{\sigma_\beta^G \mid \beta < \alpha \wedge \exists q \leq p : q \in G \wedge q \Vdash \sigma_\beta \in \sigma_\alpha\right\} \text{ (by induction)} \\ &= \sigma_\alpha^G \end{aligned}$$

In the last equality "$\subseteq$" holds: If there is a $q \leq p$, $q \in G$, $q \Vdash \sigma_\beta \in \sigma_\alpha$, then $\sigma_\beta^G \in \sigma_\alpha^G$. In the last equaliy "$\supseteq$" holds: Suppose $V[G] \models \tau^G \in \sigma_\alpha^G$, then $\tau^G = \sigma_\beta^G$ for some $\beta < \alpha$. Hence there is $q \in A_p^{\alpha,\beta}$, $q \in G$ that forces $\tau = \sigma_\beta$. □

April 20

The following result shows that we can compute the forcing relation for a forcing $\mathbb{P} \in H_\kappa$ in H_κ.

Lemma 1.2.3. *If κ is regular and $\mathbb{P} \in H_\kappa$ has the κ-c.c., then for any formula $\varphi(x_0, ..., x_n)$, any $p \in \mathbb{P}$ and any names $\sigma_0, ..., \sigma_n$ with $p \Vdash \sigma_0, ..., \sigma_n \in H_\kappa$, there are names $\dot{\sigma}_0, ..., \dot{\sigma}_n \in H_\kappa$ such that $p \Vdash \sigma_i = \dot{\sigma}_i$ for all $i \leq n$ and*

$$(p \Vdash H_\kappa \models \varphi(\sigma_0, ..., \sigma_n)) \Leftrightarrow (H_\kappa \models p \Vdash \varphi(\dot{\sigma}_0, ..., \dot{\sigma}_n)).$$

Proof. We assume that $n = 0$ and let $\sigma = \sigma_0$, $\dot{\sigma} = \dot{\sigma}_0$. We prove the claim by induction on the complexity of formulas. By Lemmas 1.2.2 and 1.2.1 we may set $\dot{\sigma} = \sigma$. The induction step for $\wedge$ is trivial.

We begin with atomic formulas. Let $\varphi(x, y) = x \in y$, since we can write $x = y$ equivalently as $\forall z : z \in x \leftrightarrow z \in y$ and H_κ satisfies Extensionality. Obviously, $p \Vdash$ "$H_\kappa \models \dot{x} \in \dot{y}$" iff $p \Vdash \dot{x} \in \dot{y}$. So it suffices to show $p \Vdash \dot{x} \in \dot{y} \Leftrightarrow H_\kappa \models p \Vdash \dot{x} \in \dot{y}$. We do an induction over the rank of $\dot{y}$: If $\mathrm{rk}(\dot{y}) = 0$, $\dot{y}$ is (a name for) the empty set, so both $p \Vdash \dot{x} \in \dot{y}$ and $H_\kappa \models p \Vdash \dot{x} \in \dot{y}$ are false. Now consider $\mathrm{rk}(\dot{y}) > 0$. Suppose $p \Vdash \dot{x} \in \dot{y}$. Then $D_{\dot{x},\dot{y}} = \{r \mid \exists (\dot{z}, q) \in \dot{y} : r \leq q \wedge r \Vdash \dot{x} = \dot{z}\}$ is dense below p. We can write $D_{\dot{x},\dot{y}}$ as $\{r \mid \exists (\dot{z}, q) \in \dot{y} : r \leq q \wedge \forall \dot{a} : (r \Vdash \dot{a} \in \dot{x}) \leftrightarrow (r \Vdash \dot{a} \in \dot{z})\}$. So we can apply the inductive hypothesis and obtain $D_{\dot{x},\dot{y}}^{H_\kappa} = D_{\dot{x},\dot{y}}$ and hence $H_\kappa \models$ "$D_{\dot{x},\dot{y}}$ is dense below p". Thus $H_\kappa \models p \Vdash \dot{x} \in \dot{y}$. The backwards direction follows since the statement is Σ_2.

Suppose that $\varphi = \neg\psi$ and that the lemma holds for ψ. For the backward direction suppose $H_\kappa \models p \Vdash \neg\psi$. If $p \Vdash \neg(H_\kappa \models \psi)$, we are done. Otherwise there is some $q \leq p$ that forces $H_\kappa \models \psi$, which by the induction hypothesis yields $H_\kappa \models q \Vdash \psi$, contradicting the assumption. The forward direction is similar.

Lastly assume $\varphi = \exists x\psi$ and that the lemma holds for ψ. Then:

$$\begin{aligned} & p \Vdash H_\kappa \models \exists x\psi(x) \\ \Leftrightarrow\ & \exists \dot{x} \in H_\kappa : p \Vdash H_\kappa \models \psi(\dot{x}) && \text{(by Lemmas 1.2.2 , 1.2.1, the max. principle)} \\ \Leftrightarrow\ & \exists \dot{x} \in H_\kappa : H_\kappa \models p \Vdash \psi(\dot{x}) && \text{(by induction hypothesis)} \\ \Leftrightarrow\ & H_\kappa \models \exists \dot{x} : p \Vdash \psi(\dot{x}) \\ \Leftrightarrow\ & H_\kappa \models p \Vdash \exists x\psi(x) && \text{(by the maximality principle).} \end{aligned}$$

□

Lemma 1.2.4. *Suppose that $\kappa > \omega_1$ is regular. Let $\mathbb{P}_\kappa$ be a countable support iteration of length κ such that all stages satisfy the κ-cc. Then $\mathbb{P}_\kappa$ satisfies the κ-cc.*

Proof. Assume $A = (p_\xi \mid \xi < \kappa)$ is an antichain in $\mathbb{P}_\kappa$. We may assume its indices have uncountable cofinality. Let $F(\xi) = \min\{\alpha \mid \mathrm{supp}(p_\xi) \cap \xi \subseteq \alpha\}$. Since $\mathbb{P}_\kappa$ has countable supports, F is regressive. By Fodor's Lemma, e.g., [?, Theorem 8.7], there is a stationary $S \subseteq \kappa$ and $\gamma < \kappa$ with $F[S] = \{\gamma\}$. Construct $\{\alpha_i \mid i \in S\} = S' \subseteq S$, $|S'| = \kappa$ with $\forall \xi < \zeta \in S' : \mathrm{supp}(p_\xi) \subseteq \zeta$ by recursion:

$$\alpha_i = \min(S \setminus (\sup_{j<i}(\mathrm{supp}(p_{\alpha_j}) \cup \alpha_j))).$$

Note that if $\xi < \zeta \in S'$, then $\mathrm{supp}(p_\xi) \subseteq \zeta$ and $\mathrm{supp}(p_\zeta) \cap \zeta \subseteq \gamma$, therefore $\mathrm{supp}(p_\xi) \cap \mathrm{supp}(p_\zeta) \subseteq \gamma$.

Since $\mathbb{P}_\gamma$ satisfies the κ-cc, there are $\xi < \zeta \in S'$ and $r' \in \mathbb{P}_\gamma$ such that $r' \leq p_\xi \upharpoonright \gamma, p_\zeta \upharpoonright \gamma$. Define a condition $q = (q(\alpha) \mid \alpha < \kappa) \in \mathbb{P}_\kappa$ by:

$$q(\alpha) = \begin{cases} r'(\alpha), \alpha < \gamma, \\ p_\xi(\alpha), \alpha \geq \gamma \wedge \alpha \in \mathrm{supp}(p_\xi), \\ p_\zeta(\alpha), \alpha \geq \gamma \wedge \alpha \in \mathrm{supp}(p_\zeta), \\ \mathbb{1}, \qquad \text{otherwise.} \end{cases}$$

This is well-defined, since above γ the supports of p_ζ and p_ξ are disjoint. But then $q \leq p_\xi$ and $q \leq p_\zeta$, i.e., A is no antichain, contradicting our assumption. □

The lemma is false for $\kappa = \omega_1$.

Exercise 1.2.5. (1) *Show that the countable support iteration of the forcing $\{p, q, 1\}$ with $p \perp q$ of length ω is not c.c.c.*
(2) *Show that any countable support iteration of nonatomic forcings of length ω is not c.c.c.*

Lemma 1.2.6. *Let M be a transitive model of* ZFC *with* $\mathrm{Ord} \subseteq M$*, $\mathbb{P} \in M$ a λ^+-cc forcing notion, G some $\mathbb{P}$-generic filter on M and λ a cardinal. In $V[G]$, if $V \models M^\lambda \subseteq M$ then $M[G]^\lambda \subseteq M[G]$.*

Proof. We work in $V[G]$. Let $c = (c_\alpha \mid \alpha < \lambda)$ be a λ-sequence such that for all $\alpha < \lambda$, $c_\alpha \in M[G]$. For each $\alpha < \lambda$, let $\dot{c_\alpha}$ be a $\mathbb{P}$-name with $\dot{c_\alpha}^G = c_\alpha$. Let $\dot{a}$ be a $\mathbb{P}$-name with $\dot{a}^G = (\dot{c_\alpha} \mid \alpha < \lambda)$. Choose a $p \in G$ with $p \Vdash \forall \alpha < \check{\lambda} : \dot{a}(\alpha) \in M^{\mathbb{P}}$ in V.

Working in V, for each $\alpha < \lambda$, there is a maximal antichain A_α below p such that every $q \in A_\alpha$ decides $\dot{a}(\alpha)$, i.e., for some $x \in M$, $q \Vdash \dot{a}(\alpha) = \check{x}$. Define $\sigma = \{(\check{(\alpha, x)}, q) \mid \alpha < \lambda, q \in A_\alpha, q \Vdash \dot{a}(\alpha) = \check{x}\}$. Then $p \Vdash \sigma = \dot{a}$. Notice that $|\sigma| \leq \lambda$, since for each α, $|A_\alpha| \leq \lambda$. Thus $\sigma \in M$.

in $V[G]$ again, $(\dot{c_\alpha} \mid \alpha < \lambda) = \dot{a}^G = \sigma^G \in M[G]$. We can compute $c = (c_\alpha \mid \alpha < \lambda) = (\dot{c_\alpha}^G \mid \alpha < \lambda)$ from $(\dot{c_\alpha} \mid \alpha < \lambda)$ and G. Hence by Replacement, $c \in M[G]$. □

Exercise 1.2.7. *Prove Lemma 1.2.6 without using names for names.*

Lemma 1.2.8. *Let λ be a cardinal and $M^\lambda \subseteq M$ for some model M with* $\mathrm{Ord} \subseteq M$*. Then $H^M_{\lambda^+} \supseteq H_{\lambda^+}$.*

Proof. Let $x \in H_{\lambda^+}$ and set $\mu := |\mathrm{tc}(\{x\})| \leq \lambda$. Find a bijection $f : |\mathrm{tc}(\{x\})| \to \mathrm{tc}(\{x\})$ with $f(\emptyset) = x$. Now define a relation R on μ by $\alpha R \beta \leftrightarrow f(\alpha) \in f(\beta)$. Then, (μ, R) has the transitive collapse $(\mathrm{tc}(\{x\}, \in)$. By assumption $M^\lambda \subseteq M$, hence $R \in M$. We can reconstruct x from R as the transitive collapse. □

Exercise 1.2.9. *Every measurable cardinal is inaccessible.*

Lemma 1.2.10. *Suppose that $\kappa \leq \lambda$ are cardinals, U is a normal ultrafilter on $P_\kappa(\lambda)$ and $j = j_U$.*

(a) *Suppose that $f, g \colon P_\kappa(\lambda) \to \kappa$.*
  (i) $[f] = [g] \iff j(f)(j[\lambda]) = j(g)(j[\lambda])$.
  (ii) $[f] \in [g] \iff j(f)(j[\lambda]) \in j(g)(j[\lambda])$.

(b) $[f] = j(f)(j[\lambda])$.

(c) $j(\kappa) > \lambda$.

Proof. (a) This follows from the definition of U.

(b) The map $\pi \colon \{[g] \mid [g] \in [f]\} \to j(f)(j[\lambda])$, $\pi([g]) = j(f)(j[\lambda])$ is an isomorphism by (a).

(c) Let $f \colon P_\kappa(\lambda) \to \kappa$, $f(x) = \mathrm{otp}(x)$. Since $[\mathrm{id}] = j[\lambda]$, $[f] = \lambda$. Moreover $[f] < [c_\kappa] = j(\kappa)$. □

Definition 1.2.11. Suppose that $\kappa \leq \lambda$ are regular uncountable cardinals.

(a) Let $[\lambda]^\kappa = P_{\kappa^+}(\lambda)$ denote the set of subsets of λ of size $\leq \kappa$.

(b) A subset S of $[\lambda]^\kappa$ is *stationary* if $S \cap C \neq \emptyset$ for every club subset C of $[\lambda]^\kappa$.

Definition 1.2.12. (a) We say that M is an elementary submodel of N if $(M, \in)$ is an elementary submodel of $(N, \in)$.

(b) Suppose that $\mathbb{P}$ is a forcing and M is an elementary submodel of H_λ for some cardinal λ a condition $q \in \mathbb{P}$ is $(M, \mathbb{P})$-*generic* (an $(M, \mathbb{P})$-*master condition*)) if for every maximal antichain $A \in M$, the set $A \cap M$ is predense below q.

Lemma 1.2.13. *Suppose that $\mathbb{P}$ is a forcing. The following conditions are equivalent.*

(a) *If λ is an uncountable regular cardinal, S is a stationary subset of $[\lambda]^\omega$ and G is $\mathbb{P}$-generic over V, then S is stationary in $V[G]$.*

(b) *$\mathbb{P}$ is* proper, *i.e. for $\lambda = (2^{|\mathbb{P}|})^+$, there is a club of elementary substructures M of H_λ such that for every $p \in M$, there is an $(M, \mathbb{P})$-generic condition $q \leq p$.*

(c) *There is some $\lambda_0 \in \mathsf{Card}$ such that for all regular $\lambda \geq \lambda_0$, there is a club of elementary substructures M of H_λ such that for every $p \in M$, there is an $(M, \mathbb{P})$-generic condition $q \leq p$.*

Proof. See [Jech, chapter 31] □ cite

1.3. **Consistency of the proper forcing axiom.** April 26

Axiom 1.3.1 (Proper Forcing Axiom (PFA)). If $(\mathbb{P}, <)$ is a proper forcing notion and $\mathcal{D}$, $|\mathcal{D}| = \aleph_1$, is a collection of predense subsets of $\mathbb{P}$, then there exists a $\mathcal{D}$-generic filter on $\mathbb{P}$.

Axiom 1.3.2 (Bounded Fragments of PFA). Let λ be a cardinal.

(i) *PFA_λ* is the following axiom: Let $(\mathbb{P}, <)$ be a proper *partial order* and $\mathcal{D}$, $|\mathcal{D}| = \aleph_1$ be collection of predense subsets of $\mathbb{P}$ such that for all $D \in \mathcal{D}$, $|D| \leq \lambda$. Then there exists a $\mathcal{D}$-generic filter on $\mathbb{P}$.

(ii) A *counterexample to PFA_λ* is a proper partial order $(\mathbb{P}, <)$ such that there is a collection $\mathcal{D}$ with $|\mathcal{D}| = \aleph_1$ of predense subsets of $\mathbb{P}$ such that for all $D \in \mathcal{D}$, $|D| \leq \lambda$ and there exists no $\mathcal{D}$-generic filter on $\mathbb{P}$.

Definition 1.3.3. Suppose that $\{\mathbb{P}_\alpha \mid \alpha < \lambda\}$ is a set of forcing notions. The *lottery sum* of the $\mathbb{P}_\alpha$ is their disjoint union $\mathbb{P}$ with a new $\mathbb{1}$ such that $\mathbb{1} > p$ for all $p \in P_\alpha$, $\alpha < \lambda$.

Lemma 1.3.4. *Lottery sums of proper forcings are themselves proper.*

Proof. Let $\mathbb{P}$ be the lottery sum of $(\mathbb{Q}_\alpha \mid \alpha < \kappa)$. Let G be $\mathbb{P}$-generic. Since elements of G are pairwise compatible and if $p, q \in \mathbb{P}$, $p \in \mathbb{Q}_\alpha$, $q \in \mathbb{Q}_\beta$, $\alpha \neq \beta$, p, q are incompatible, $G \subseteq \mathbb{Q}_\alpha \cup \{\mathbb{1}\}$ for some α. A set D is clearly dense in $\mathbb{P}$ if and only if $D \cap \mathbb{Q}_\alpha$ is dense in $\mathbb{Q}_\alpha$ for all $\alpha < \kappa$. Hence G is a $\mathbb{Q}_\alpha$-generic filter for some α, i.e., stationary sets of $[\lambda]^\omega$ for regular uncountable cardinals λ are preserved between V and $V[G]$. □

Definition 1.3.5. Suppose that C is a class. An element x of C is *hereditarily minimal in* C if $|\mathrm{tc}(x)| \leq |\mathrm{tc}(y)|$ for all $y \in C$. The *hereditary size* of x is $|\mathrm{tc}(x)|$.

We can now define a general scheme for the iterations which we will use.

Definition 1.3.6. Suppose that κ, λ are cardinals with $\omega < \lambda < \kappa$. The *minimal counterexample iteration* $\mathbb{P}_\kappa = \mathbb{P}_\kappa^{PFA_\lambda}$ for *PFA* of length κ is the countable support iteration of $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \kappa,\ \beta < \kappa)$, where $\mathbb{P}_\alpha$ and $\dot{\mathbb{Q}}_\alpha$ are defined by induction: Let $\vec{\mathbb{Q}} = \langle \dot{\mathbb{Q}}_\beta \mid \beta < \lambda \rangle$ be an enumeration of all names $\dot{\mathbb{Q}}$ of minimal hereditary size smaller than κ such that $\mathbb{1}_{\mathbb{P}} \Vdash \dot{\mathbb{Q}}$ is a counterexample to PFA_λ of minimal hereditary size smaller than κ. Let $\dot{\mathbb{Q}}_\alpha$ be the canonical $\mathbb{P}_\alpha$-name for the lottery sum of $\vec{\mathbb{Q}}$.

We will only consider iterations of inaccessible length κ.

Lemma 1.3.7. *If κ is inaccessible and $\alpha < \kappa$, then $|\mathbb{P}_\alpha| < \kappa$.*

Proof. This is shown by induction on α. If $\alpha = 0$, then $\mathbb{P}_\alpha$ is a union of forcings of hereditary size $\gamma < \kappa$, so $\mathbb{P}_\alpha \subseteq H_{\gamma^+}$. Therefore $|\mathbb{P}_\alpha| \leq |H_{\gamma^+}| \leq 2^\gamma < \kappa$.

If $\alpha = \beta + 1$, then $\mathbb{P}_\beta$ forces that $\dot{\mathbb{Q}}_\alpha$ is a union of forcing notions with hereditary size $\gamma < \kappa$, so exactly as above, $\mathbb{1}_\beta \Vdash \left|\dot{\mathbb{Q}}_\alpha\right| \leq |H_{\gamma^+}| \leq 2^\gamma$. Hence $\mathbb{P}_\alpha \leq |\mathbb{P}_\beta| \cdot 2^\gamma < \kappa$.

Suppose that $\gamma < \kappa$ is a limit and that for all $\alpha < \gamma$, $|\mathbb{P}_\alpha| < \kappa$. Since κ is regular, there is some λ such that for all $\alpha < \gamma$, $|\mathbb{P}_\alpha| < \lambda$. We have $|\mathbb{P}_\gamma| \leq \Pi_{\alpha<\gamma} |\mathbb{P}_\alpha|$, since $p \mapsto (p \upharpoonright \alpha)_{\alpha<\gamma}$ is injective. Hence $\Pi_{\alpha<\gamma} |\mathbb{P}_\alpha| \leq \Pi_{\alpha<\gamma} \lambda = \lambda^\gamma < \kappa$. □

$\mathbb{P}_\kappa$ is absolute between transitive models M of ZFC that contain H_κ as a subset, by the following lemma.

Lemma 1.3.8. *Suppose that κ is inaccessible. If M is transitive with $H_\kappa \subseteq M$, then $\mathbb{P}_\kappa^M = \mathbb{P}_\kappa$.*

Proof. The point is that if $\mathbb{P}$ is a forcing in H_κ, then it is proper if and only if it is proper in H_κ. Using this, we will show that the definition of the sequence $(\mathbb{P}_\alpha \mid \alpha < \kappa)$ is absolute between H_κ and V, where the $\mathbb{P}_\alpha$ are the initial segments of $\mathbb{P}_\kappa$.

If γ is a limit and $\mathbb{P}_\alpha = \mathbb{P}_\alpha^M$ for all $\alpha < \gamma$, then $\mathbb{P}_\gamma = \mathbb{P}_\gamma^M$. Suppose that $\alpha = \beta + 1$ and $\mathbb{P}_\beta^M = \mathbb{P}_\beta$. We need to show that $\dot{\mathbb{Q}}_\alpha^M = \dot{\mathbb{Q}}_\alpha$.

Claim 1.3.9. Suppose that $\dot{Q}$ is a $\mathbb{P}_\alpha$-name for a forcing. Then $p \Vdash_{\mathbb{P}_\alpha} \dot{Q}$ is proper $\iff$ $H_\kappa \models (p \Vdash_{\mathbb{P}_\alpha} \dot{Q}$ is proper$) \iff M \models (p \Vdash_{\mathbb{P}_\alpha} \dot{Q}$ is proper$)$.

Proof. This follows from Lemma 1.2.3 and the definition of properness, since κ is inaccessible. □

This implies that $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{Q}}_\alpha^{H_\kappa} = \dot{\mathbb{Q}}_\alpha^M$. □

Theorem 1.3.10. *If κ is λ-supercompact, then $\mathbb{P}_\kappa^{\mathrm{PFA}}$ forces that* PFA *holds for all proper forcings $\mathbb{P}$ with $2^{|\mathbb{P}|} \leq \lambda$.*

Proof. Let $j \colon V \to M$ be a λ-supercompact embedding with $\mathrm{crit}(j) = \kappa$, $\lambda < j(\kappa)$, $M^\lambda \subseteq M$.

Suppose that $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \kappa \rangle$ is the iteration defined above. Suppose that $\dot{\mathbb{Q}}$ and $\dot{\mathcal{D}}$ are $\mathbb{P}_\kappa$-names and $p_0 \in \mathbb{P}_\kappa$ forces that $\dot{\mathbb{Q}}$ is a counterexample to PFA of minimal hereditary size with $2^{|\dot{\mathbb{Q}}|} \leq \lambda$, $\dot{\mathcal{D}}$ is a sequence of length ω_1 of open dense subsets of $\dot{\mathbb{Q}}$ and there is no $\dot{\mathcal{D}}$-generic centered set. Moreover, suppose that $\dot{\mathbb{Q}}$ is of minimal hereditary size.

Since $\mathbb{P}_\alpha \in H_\kappa \subseteq M$ for all $\alpha < \kappa$, $\mathbb{P}_\kappa \subseteq M$. Moreover $j(\mathbb{P}_\alpha) = \mathbb{P}_\alpha$ for all $\alpha < \kappa$, since $j \upharpoonright H_\kappa = \mathrm{id}$. In M, the forcing $j(\mathbb{P}_\kappa)$ is, by elementarity, a countable support iteration of length $j(\kappa) > \lambda$ and $\mathbb{P}_\kappa$ is an initial segment of $j(\mathbb{P}_\kappa)$, since $\mathrm{crit}(j) = \kappa$.

Suppose that H is $j(\mathbb{P}_\kappa)$-generic over V with $j(p_0) \in H$. Then H is $j(\mathbb{P}_\kappa)$-generic over M. We work in $V[H]$. Let $H_{<\kappa}$ denote the restriction of H to $\mathbb{P}_{<\kappa}$. Then $H_{<\kappa}$ is $\mathbb{P}_\kappa$-generic over V. Let H_κ denote the restriction of H to $\dot{\mathbb{Q}}^{H_{<\kappa}}$. Then H_κ is $\dot{\mathbb{Q}}^{H_{<\kappa}}$-generic over $V[G_{<\kappa}]$.

Let $G = H_{<\kappa}$, $\mathbb{P} = \dot{\mathbb{Q}}^{H_{<\kappa}}$, $\mathcal{D} = (D_\alpha \mid \alpha < \omega_1) = \dot{\mathcal{D}}^G$. Then $\mathbb{P} \in M[G]$ by Lemma 1.2.8.

Claim 1.3.11. In $M[G]$, $\mathbb{P}$ violates *PFA*, is of minimal hereditary size with that property and $\mathbb{P} \in H_{j(\kappa)}$.

Proof. We first claim that $|\mathrm{tc}(\mathbb{P})| = |\mathbb{P}|$. Otherwise, take a bijection $f : \mathbb{P} \to \alpha = |\mathbb{P}|$ and define a relation $<_\alpha$ on α by $\beta <_\alpha \gamma$ iff $f^{-1}(\beta) <_{\mathbb{P}} f^{-1}(\gamma)$. $(\alpha, <_\alpha)$ is a forcing notion equivalent to $\mathbb{P}$ but of smaller hereditary size $\mathrm{tc}(\alpha) = \alpha$, contradicting the assumption.

We now show that $\mathbb{P}$ is proper in $M[G]$. Let $\mu = (|\mathbb{P}|)^+$. Since we now know $|\mathrm{tc}(\mathbb{P})| = |\mathbb{P}| < \mu$, $\mathbb{P} \in H_\mu$. Choose a club $C \subseteq [H_\mu]^\omega$ witnessing that $\mathbb{P}$ is proper in $V[G]$. Note that

$$|C| \leq |H_\mu| \leq 2^{<\mu} = 2^{|\mathbb{P}|} \leq \lambda$$

and therefore by Lemma 1.2.8, $C \in M[G]$ and hence C witnesses that $\mathbb{P}$ is proper in $M[G]$.

Also, $V[G]$ and $M[G]$ have the same $\aleph_1$, since $\mathbb{P}_\kappa$ is proper (as a countable support iteration of proper forcing notions). Hence, $|\mathcal{D}|^{M[G]} = \aleph_1^{M[G]}$. For all $\alpha < \omega_1$, $D_\alpha \subseteq \mathbb{P} \in M[G]$, $|D_\alpha| \leq |\mathbb{P}| \leq \lambda$, i.e., $D_\alpha \in M[G]$. Thus, since $\aleph_1 < \lambda$, $\mathcal{D} \in M[G]$.

Furthermore, $|\mathrm{tc}(\mathbb{P})| < \lambda < j(\kappa)$, so $\mathbb{P} \in H_{j(\kappa)}$. Finally, if there were a hereditary smaller counterexample in $M[G]$, it would be in $V[G]$ and be a counterexample to PFA there, because $M[G]$ is sufficiently closed to contain filters witnessing the contrary and clubs witnessing properness. Hence this would contradict the hereditarily minimality of $\mathbb{P}$. $\square$

We now work in $V[H]$. We define j^* as follows. April 27

$$j^* \colon V[G] \to M[H],$$
$$j^*(\sigma^G) = j(\sigma)^H.$$

Claim 1.3.12. j^* is well-defined and elementary and extends j.

Proof. To show that j^* is well-defined, let σ, τ be $\mathbb{P}_\kappa$-names with $\sigma^G = \tau^G$. Then there is $p \in G$ such that $p \Vdash \sigma = \tau$, i.e., $j(p) \Vdash j(\sigma) = j(\tau)$.

Suppose that $p = (p_\alpha \mid \alpha < \kappa)$. Then there is some $\beta < \kappa$ with $p_\gamma = \mathbb{1}$ for all γ with $\beta \leq \gamma < \kappa$. Since $\mathrm{crit}(j) = \kappa$, $j(p)(\gamma) = \mathbb{1}$ for all γ with $\beta \leq \gamma < j(\kappa)$. Hence $j(p) \in H$.

To show that j^* is elementary, let $\varphi = \varphi(x)$ be a formula, σ a $\mathbb{P}_\kappa$-name and suppose that $V[G] \models \varphi(\sigma^G)$. Then there is some $p \in G$ with $p \Vdash \varphi(\sigma)$, i.e., $j(p) \Vdash \varphi(j(\sigma))$. As above $j(p) \in H$.

Moreover j^* extends j, since $j^*(x) = j^*(\check{x}^G) = j(\check{x})^H = \check{j(x)}^H = j(x)$ for $x \in V$. $\square$

As in (i), $\mathcal{D}$ is a family of size $\aleph_1$ of dense subsets of $\mathbb{P}$ in $M[H]$. We show that there is a $(j^*(\mathbb{P}), j^*(\mathcal{D}))$-generic filter in $M[H]$. Notice that $j^* \upharpoonright \mathbb{P} \in M[H]$ by Lemma 1.2.6, since $|\mathbb{P}| < \lambda$. $G_\kappa \subseteq \mathbb{P}$ and therefore by Replacement $j^*[G_\kappa] \in M[H]$.

Since $j^*(\omega_1) = \omega_1$, $j^*(\mathcal{D}) = \{j^*(D) \mid D \in \mathcal{D}\}$. Since G_κ is $\mathbb{P}$-generic over $V[G]$, it intersects every $D \in \mathcal{D}$. Thus for every $D \in \mathcal{D}$ there is some $x_D \in G_\kappa$ such that $V[G] \models x_D \in D$, so by elementarity, $M[H] \models j^*(x_D) \in j^*(D)$.

Therefore the filter on $j^*(\mathbb{P})$ generated by $j^*[G_\kappa]$ in $M[H]$ intersects every $D \in j^*(\mathcal{D})$. Hence, by elementarity, there is a filter on $\mathbb{P}$ in $V[G]$ which intersects every $D \in \mathcal{D}$. $\square$

The classical result follows immediately.

Corollary 1.3.13. *If κ is a supercompact cardinal, then $\mathbb{1}_{\mathbb{P}_\kappa}$ forces* PFA*. Hence* PFA *is consistent relative to the existence of a supercompact cardinal.*

Definition 1.3.14. Suppose that κ is a cardinal and $\mathbb{P}$ is a forcing.

(a) $\mathbb{P}$ is $< \kappa$-closed if for every strictly decreasing sequence $\langle p_\alpha \mid \alpha < \gamma\rangle$ with $\gamma < \kappa$, there is some $p \in \mathbb{P}$ such that for all $\alpha < \gamma$, $p \leq p_\alpha$.
(b) A set $C \subseteq \mathbb{P}$ is *directed* iff for all $a, b \in C$ there is $c \in C$ with $c \leq a, b$.
(c) $\mathbb{P}$ is $< \kappa$-directed closed if for every directed subset C of $\mathbb{P}$ with $|C| < \kappa$, there is some $p \in \mathbb{P}$ such that for all $q \in C$, $p \leq q$.

Theorem 1.3.15 (Paul Larson)**.** *PFA is preserved by $< \omega_2$-directed closed forcing.*

Proof. Suppose that $\mathbb{P}$ is $< \omega_2$-directed closed. Suppose that $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name and $\mathbb{1}_{\mathbb{P}}$ forces that $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a proper forcing. Then $\mathbb{P} * \dot{\mathbb{Q}}$ is proper. Suppose that $\dot{\mathcal{D}}$ is a $\mathbb{P}$-name for a sequence of length ω_1 of open dense subsets of $\mathbb{P}$. Since $\mathbb{P} * \dot{\mathbb{Q}}$ is proper and hence preserves ω_1, there is a sequence $\langle \dot{D}_\alpha \mid \alpha < \omega_1\rangle$ of $\mathbb{P}$-names such that $\mathbb{1}_{\mathbb{P}}$ forces that $\dot{\mathcal{D}} = \langle \dot{D}_\alpha \mid \alpha < \omega_1\rangle$.

Let $D_\alpha = \{(p, \dot{q}) \mid p \Vdash_{\mathbb{P}} \dot{q} \in \dot{D}_\alpha\}$ for $\alpha < \omega_1$. Since $\dot{D}_\alpha$ is name for an open dense set, D_α is open dense for each $\alpha < \omega_1$.

Suppose that $p_0 \in \mathbb{P}$. By PFA applied to $\mathbb{P}/p_0 = \{q \in \mathbb{P} \mid q \leq p_0\}$, there is a filter G in $\mathbb{P}/p_0$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$. Let $\bar{D} = \bigcup_{\alpha<\omega_1} D_\alpha$.

Claim 1.3.16. $G \cap \bar{D}$ is directed.

Proof. Suppose that $p, q \in G \cap \bar{D}$. Suppose that $p \in D_\alpha$ and $q \in D_\beta$. Since G is a filter, there is some $r \leq p, q$ in G. Then $r \in G \cap D_\alpha \cap D_\beta \subseteq G \cap \bar{D}$. $\square$

Claim 1.3.17. There is a directed subset F of $G \cap \bar{D}$ of size ω_1 such that for all $\alpha < \omega_1$, $G \cap D_\alpha \neq \emptyset$.

Proof. We construct a sequence $\langle F_n \mid n \in \omega\rangle$ such that $|F_n| = \omega_1$ for all $n \in \omega$. We choose a subset F_0 of $G \cap \bar{D}$ such that $F_0 \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$. Suppose that F_n is defined. We choose a subset F_{n+1} of $G \cap \bar{D}$ such that $F_n \subseteq F_{n+1}$ and for all $p, q \in F_n$, there is some $r \leq p, q$ in F_{n+1}. Let $F = \bigcup_{n\in\omega} F_n$. $\square$

Since F is directed, there is a condition $p_1 \leq p_0$ with $p_1 \leq q$ for all $q \in F$. Let $\dot{H} = \{(\dot{q}, r) \mid \exists \dot{q} \mid (r, \dot{q}) \in F\}$.

Claim 1.3.18. p_1 forces that $\dot{H}$ is directed.

Proof. Suppose that G is $\mathbb{P}$-generic over V. Suppose that $(\dot{q}, r), (\dot{t}, s) \in \dot{H}$ and $r, s \in G$. Since F is directed, there is some $(u, \dot{v}) \in F$ with $(u, \dot{v}) \leq (r, \dot{q}), (s, \dot{t})$. Since $p_1 \leq u$, $p_1 \Vdash \dot{v} \leq \dot{q}, \dot{t}$ and $p_1 \Vdash \dot{v} \in \dot{H}$. $\square$

Claim 1.3.19. p_1 forces that for all $\alpha < \omega_1$, $\dot{H} \cap \dot{D}_\alpha \neq \emptyset$.

Proof. Suppose that $(s, \dot{t}) \in F \cap D_\alpha$. Since $p_1 \leq s$, $p_1 \Vdash_{\mathbb{P}} \dot{t} \in \dot{H} \cap \dot{D}_\alpha$. $\square$

The upwards closure of a directed set is a filter. Hence p_1 forces that there is a $\dot{\mathcal{D}}$-generic filter. Since p_0 was arbitrary, $\mathbb{1}_{\mathbb{P}}$ forces that there is a $\dot{\mathcal{D}}$-generic filter. $\square$

1.4. **Axiom A forcings of size continuum.** We prove the consistency of the proper forcing axiom restricted to Axiom A forcings of size continuum relative to a weakly compact cardinal. This is a result of Baumgartner.

cite

Definition 1.4.1. A κ-model is a transitive model M of ZFC^- (i.e. ZFC without the power set axiom) of size κ such that $\kappa \subseteq M$, $\kappa \in M$ and $M^{<\kappa} \subseteq M$.

Definition 1.4.2. (a) A filter F on κ is *uniform* if for all $\alpha < \kappa$, $[\alpha, \kappa) \in F$.

(b) An inaccessible cardinal κ has the *filter property* if for all $X \subseteq P(\kappa)$ of size κ, there is a $< \kappa$-complete uniform filter F such that for every $A \in X$, $A \in F$ or $\kappa \setminus A \in F$.

Lemma 1.4.3. *Suppose that κ is inaccessible. The following conditions are equivalent.*

(a) *κ is weakly compact, i.e. $\kappa \to (\kappa)^2_2$.*
(b) *κ has the* tree property*, i.e. every κ-tree T has a cofinal branch.*
(c) *κ has the filter property.*
(d) *For every κ-model M, there is an elementary embedding $j\colon M \to N$ into a transitive model N with $\mathrm{crit}(j) = \kappa$.*
(e) *For every κ-model M, there is an elementary embedding $j\colon M \to N$ into a transitive model N with $\mathrm{crit}(j) = \kappa$ and $j, M \in N$ (this is called the* Hauser property*) .*

Proof. The equivalence of (a) and (b) was proved in models of set theory.

(b)$\Rightarrow$(c) Suppose that $X = \{A_\alpha \mid \alpha < \kappa\}$. We define $A^i_\alpha = A_\alpha$ if $i = 0$ and $A^i_\alpha = \kappa \setminus A_\alpha$ if $i = 1$. Let $A_t = \bigcap_{t(\alpha)=i} A^i_\alpha$ for $t \in 2^{<\kappa}$. We define $T = \{t \in 2^{<\kappa} \mid |A_t| = \kappa\}$.

Claim 1.4.4. $\mathrm{ht}(T) = \kappa$.

Proof. For all $\alpha < \kappa$, $\kappa = \bigcup_{t \in 2^\alpha} A_t$, since for each $\beta < \kappa$ we can choose $t \in 2^\alpha$ with $t(\bar\alpha) = 0$ if $\beta \in A_{\bar\alpha}$ and $t(\bar\alpha) = 0$ otherwise for $\bar\alpha < \alpha$. Then $\beta \in A_t$.

Since κ is inaccessible and hence $2^\alpha < \kappa$, there is some $t \in 2^\alpha$ with $|A_t| = \kappa$. □

The tree property inplies that there is some $b \in [T]$. Let $F = \{Y \subseteq \kappa \mid \exists \alpha, \beta < \kappa\ A_{b \restriction \alpha} \cap [\beta, \kappa) \subseteq Y\}$.

(c)$\Rightarrow$(d) By Los' theorem, the ultrapower embedding $j_F\colon M \to \mathrm{ult}(M, F)$ is elementary. Since F is $< \kappa$-complete, $\mathrm{ult}(M, F)$ is well-founded and $\mathrm{crit}(j_F) = \kappa$ as in the results about ultrafilters.

(d)$\Rightarrow$(e) Suppose that M is a κ-model. There is a κ-model M' such that $M \in M'$ and $M' \models |M| = \kappa$, for instance a Skolem hull of $M \cup \{M\}$ in H_{κ^+}. By (d), there is an elementary embedding $j\colon M' \to N'$ into a transitive model N' with $\mathrm{crit}(j) = \kappa$. Suppose that $f\colon \kappa \to M$ is an enumeration of M in M'. Then $j(f)\colon j(\kappa) \to j(M)$ is an enumeration of $j(M)$ in N'. Since $\mathrm{crit}(j) = \kappa$, $j(f) \restriction \kappa$ is an enumeration of $j[M]$ in N'. We can define $j \restriction M$ from M and $j[M]$, hence $j \restriction M \in N'$. Since j is elementary and $M^{<\kappa} \subseteq M$, N is closed under $< j(\kappa)$-sequences in N'. Hence $M, j \restriction M \in N$. May 3

(e)$\Rightarrow$(b) Suppose that $(T, <_T)$ is a κ-Aronszajn tree. We can assume that $T = \kappa$. There is a κ-model M with $(M, \in) \prec (H_{\kappa^+}, \in)$, for instance a Skolem hull. Suppose that $j\colon M \to N$ is an elementary embedding into a transitive model N with $\mathrm{crit}(j) = \kappa$.

In N, $j(T) = (j(\kappa), j(<_T))$ is a $j(\kappa)$-Aronszajn tree of height $\mathrm{ht}(j(T)) = j(\kappa) > \kappa$. Then $<_T = j(<_T) \cap \kappa$. Let $T(\alpha) = \{s \in T \mid \mathrm{lh}_T(s) = \alpha\}$ denote the α-th level of T.

There is some $\alpha \in j(T)(\kappa)$. We claim that the set of its predecessors $\mathrm{pred}_{j(T)}(\alpha)$ is a branch in T.

Claim 1.4.5. For every $\alpha < \kappa$, $j(T)(\alpha) = T(\alpha)$.

Proof. If $\beta \in T(\alpha)$, then $\beta = j(\beta) \in j(T)(j(\alpha)) = T(\alpha)$.

Suppose that $\beta \in j(T)(\alpha) = j(T)(j(\alpha))$. Let $\gamma = \sup\{s \in T \mid s \in T(\alpha)\}$. Since T is a κ-tree, $\gamma < \kappa$. Then $\sup\{s \in j(T) \mid s \in j(T)(\alpha)\} = j(\gamma) = \gamma < \kappa$. Hence $\beta \leq \gamma < \kappa$. Since $\beta = j(\beta) \in j(T)(\alpha)$, $\beta \in T(\alpha)$. □

This contradicts the assumption that T is a κ-Aronszajn tree. $\square$

The following type of forcing was defined by Baumgartner before proper forcing was defined by Shelah. It implies properness, and many important proper forcings satisfy Axiom A.

Definition 1.4.6. A forcing $\mathbb{P}$ is satisfies Axiom A if there is a sequence $\langle \leq_n \mid n \in \omega \rangle$ of partial orders on $\mathbb{P}$ with the following properties.

(a) $p \leq_0 q \Rightarrow p \leq q$ and $p \leq_{n+1} q \Rightarrow p \leq_n q$ for all $n \in \omega$.
(b) if $\langle p_n \mid n \in \omega \rangle$ is a sequence with $p_0 \geq_0 p_1 \geq_1 p_2 \ldots$ then there is a condition q such that $q \leq_n p_n$ for all n.
(c) If $p \in \mathbb{P}$, $A \subseteq \mathbb{P}$ is a maximal antichain below p and $n < \omega$, then there is a $q \leq_n p$ such that
$$|\{a \mid a \in A \wedge a \text{ and } q \text{ are compatible}\}| \leq \omega.$$

Now we can state the axiom we are interested in.

Definition 1.4.7. (a) The Axiom A Forcing Axiom *AAFA* is the restriction of *PFA* to Axiom A forcings.
(b) *AAFA($\mathfrak{c}$)* is the restriction of *AAFA* to forcings of size $\leq \mathfrak{c} = 2^\omega$.

Lemma 1.4.8. *Every ccc forcing is Axiom A.*

Proof. Let $\leq_n$ be equality for all $n \in \omega$. $\square$

Lemma 1.4.9. *Every σ-closed forcing is Axiom A.*

Proof. Let $\leq_n$ be $\leq$ for all $n \in \omega$. $\square$

Lemma 1.4.10. *Suppose that ϑ is an uncountable cardinal and $M \prec H_\vartheta$. If $A \in M$ and $|A| \leq \omega$, then $A \subseteq M$.*

Proof. Since $M \vDash |A| \leq \omega$, there is a surjective function $f \colon \omega \to A$ with $f \in M$. Since $\omega \subseteq M$, $\operatorname{ran}(f) = A \subseteq M$. $\square$

Lemma 1.4.11. *Every Axiom A forcing is proper.*

Proof. Suppose that $\vartheta \geq (2^{|\mathbb{P}|})^+$ is a regular cardinal. Suppose that $M \prec H_\vartheta$ is countable with $\mathbb{P} \in M$ and $p \in \mathbb{P}$.

We define a decreasing sequence $\langle p_n \mid n \in \omega \rangle$ with $p_{n+1} \leq_n p_n$ for all $n \in \omega$. Suppose that $\langle A_n \mid 1 \leq n < \omega \rangle$ enumerates the set of all maximal antichains $A \in M$.

Let $p_0 = p$. If p_n is defined, find some $p_{n+1} \leq_n p_n$ such that $|\{a \in A_n \mid a \parallel p_{n+1}\}| \leq \omega$. Let $q \leq p_n$ for all $n \in \omega$.

Claim 1.4.12. q is $(M, \mathbb{P})$-generic.

Proof. For every $n \in \omega$, the set $A_n^q = \{a \in A_n \mid a \parallel q\}$ is predense below q, since A_n is predense. Since $|A_n^q| \leq |\{a \in A_n \mid a \parallel p_{n+1}\}| \leq \omega$, $A_n^q \subseteq M$ by Lemma 1.4.10. $\square$

This completes the proof. $\square$

Lemma 1.4.13. *The following conditions are equivalent.*

(a) *Condition 1.4 in Axiom A.*
(b) *If $p \Vdash_{\mathbb{P}} \dot{\alpha} \in \mathrm{Ord}$ and $n \in \omega$, then there is some $q \leq_n p$ and a countable set C of ordinals with $q \Vdash \dot{\alpha} \in \check{C}$.*

add later

Proof. See lecture notes. $\square$

Lemma 1.4.14. *Suppose that $\mathbb{P}$ satisfies Axiom A and $\mathbb{1}_{\mathbb{P}}$ forces that $\dot{\mathbb{Q}}$ satisfies Axiom A. Then $\mathbb{P} * \dot{\mathbb{Q}}$ satisfies Axiom A.*

Proof. Suppose that $\langle \leq_n \mid n \in \omega\rangle$ witnesses that $\mathbb{P}$ satisfies Axiom A. Suppose that $\mathbb{1}_{\mathbb{P}}$ forces that $\langle \dot{\leq}_n \mid n \in \omega\rangle$ witnesses that $\dot{\mathbb{Q}}$ satisfies Axiom A. We define $(p, \dot{q}) \leq_n (r, \dot{s})$ as $p \leq_n q$ and $p \Vdash_{\mathbb{P}} \dot{q} \dot{\leq}_n \dot{s}$.

Suppose that $\langle (p_n, \dot{q}_n) \mid n \in \omega\rangle$ is a sequence with $(p_{n+1}, \dot{q}_{n+1}) \leq_n (p_n, \dot{q}_n)$ for all $n \in \omega$. Find $p \in \mathbb{P}$ such that $p \leq_n p_n$ for all $n \in \omega$. Find a $\mathbb{P}$-name $\dot{q}$ such that $p \Vdash_{\mathbb{P}} \dot{q} \leq \dot{q}_n$ for all $n \in \omega$. Then $(p, \dot{q}) \leq_n (p_n, \dot{q}_n)$ for all $n \in \omega$.

For condition in Axiom A, use Lemma 1.4.13. □ add later

We use the following variation of the iteration in Definition 1.3.6. May 4

Definition 1.4.15. The forcing $\mathbb{P}_\kappa^{\mathrm{AAFA}}$ is defined by modifying Definition 1.3.6 by only using names for Axiom A forcings and adding the forcings $\mathrm{Add}(\omega, 1)$ and $\mathrm{Col}(\omega_1, \alpha)$ to the sequence $\vec{\mathbb{Q}} = \langle \dot{\mathbb{Q}}_\beta \mid \beta < \lambda\rangle$ of minimal counterexamples in step α for all α with $\omega_1 \leq \alpha < \kappa$.

Theorem 1.4.16. *If κ is weakly compact, then $\mathbb{P}_\kappa^{\mathrm{AAFA}}$, forces $\mathrm{AAFA}(\mathfrak{c})$ with $\mathfrak{c} = \aleph_2$.*

Proof. Suppose that $\mathbb{P}_\kappa = \mathbb{P}_\kappa^{\mathrm{AAFA}}$ and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \kappa\rangle$ is the iteration defined above. $\mathbb{P}_\kappa$ has the κ-cc by Lemma 1.2.4.

Suppose the theorem is false. Let $p \in \mathbb{P}_\kappa$ such that p_0 forces that $(\dot{\mathbb{Q}}, \dot{\mathcal{D}})$ is a hereditarily minimal counterexample to $\mathrm{AAFA}(\mathfrak{c})$ Let $\dot{\mathcal{A}}$ be a name for a sequence of partial orders on $\dot{\mathbb{Q}}$ witnessing Axiom A.

Claim 1.4.17. $\mathbb{1}_{\mathbb{P}}$ forces that $\mathfrak{c} = \kappa = \aleph_2$.

Proof. We leave this as an exercise. □

We can assume that $p_0 \Vdash \dot{\mathbb{Q}}, \dot{\mathcal{A}} \subseteq \kappa$. Thus by Lemma 1.2.2 we can suppose that $\dot{\mathbb{Q}}, \dot{\mathcal{A}} \subseteq H_\kappa$. Let λ be regular and large enough such that H_λ knows that $\dot{\mathbb{Q}}$ is a name for an Axiom A forcing as witnessed by $\dot{\mathcal{A}}$. Let $X \prec H_\lambda$ with $H_\kappa \subseteq X$, $\mathbb{P}_\kappa, \dot{\mathbb{Q}}, \dot{\mathcal{D}}, \dot{\mathcal{A}}, \kappa \in X$, $X^{<\kappa} \subseteq X$ and $|X| = \kappa$.

Let $X \to M$ be the Mostowski collapse of X, then M is a κ-model. Notice that since $\dot{\mathbb{Q}} \subseteq H_\kappa \subseteq X$ is in the transitive part of X, $\pi(\dot{\mathbb{Q}}) = \dot{\mathbb{Q}} \in M$. Likewise $\dot{\mathcal{D}} \in M$, $\dot{\mathcal{A}} \in M$ and $\mathbb{P}_\kappa \in M$. Now let $j : M \to N$ be a weak compactness embedding for κ with the Hauser property as in Lemma 1.4.3 (e).

Claim 1.4.18. If $\bar{G}$ is $\mathbb{P}_\kappa$-generic over V, then $\dot{\mathcal{A}}^{\bar{G}} = \mathcal{A} = (\leq_n \mid n \in \omega)$ witnesses that $\mathbb{Q}$ satisfies Axiom A in $N[\bar{G}]$.

Proof. We have $M[\bar{G}] \in N[\bar{G}]$ by the Hauser property and since $N[\bar{G}]$ is transitive it contains all the sets we require (since we put them in $M[\bar{G}]$). (a) and (b) in Definition 1.4.6 are clear. For 1.4, Let p, n, A be as required. Then, in $V[\bar{G}]$, there is some $q \leq_n p$ such that $|\{a \mid a \in A,\ a \text{ and } q \text{ are compatible}\}| \leq \omega$. Since $N[\bar{G}]$ and $V[\bar{G}]$ agree on $\aleph_1$ (since $\mathbb{P}_\kappa$ is proper) and on the computation of that set (since $\mathbb{Q} \subseteq N[\bar{G}]$), this is also true in $N[\bar{G}]$. □

Claim 1.4.19. If $\bar{G}$ is $\mathbb{P}_\kappa$-generic over V, then $\mathbb{Q}$ appears in the lottery sum in step κ in $\mathbb{P}_{j(\kappa)}$ in $N[\bar{G}]$.

Proof. Since $\mathbb{P}_\kappa$ has the κ-cc, we have $H_\kappa^{V[\bar{G}]} = H_\kappa^{N[\bar{G}]}$ by Lemma 1.2.1 and Lemma 1.2.2. Hence in $N[\bar{G}]$, there is no counterexample to $\mathrm{AAFA}(\mathfrak{c})$ that is smaller than $|\mathbb{Q}|$. Moreover $|\mathrm{tc}(\mathbb{Q})| < \kappa^+ \leq j(\kappa)$ and $\mathbb{Q}$ satisfies Axiom A by the previous claim. □

Since $\mathbb{P}_\alpha \in H_\kappa \subseteq M$ for all $\alpha < \kappa$, $\mathbb{P}_\kappa \subseteq M$. Moreover $j(\mathbb{P}_\alpha) = \mathbb{P}_\alpha$ for all $\alpha < \kappa$, since $j \restriction H_\kappa = \mathrm{id}$. In M, the forcing $j(\mathbb{P}_\kappa)$ is, by elementarity, a countable support iteration of length $j(\kappa) > \kappa$ and $\mathbb{P}_\kappa$ is an initial segment of $j(\mathbb{P}_\kappa)$, since $\mathrm{crit}(j) = \kappa$.

Let H be $j(\mathbb{P}_\kappa)$-generic over V with $j(p_0) \in H$. Then H is $j(\mathbb{P}_\kappa)$-generic over N. We work in $V[H]$. Let $H_{<\kappa}$ denote the restriction of H to $\mathbb{P}_{<\kappa}$. Then $H_{<\kappa}$ is $\mathbb{P}_\kappa$-generic over V. Let H_κ denote the restriction of H to $\dot{\mathbb{Q}}^{H_{<\kappa}}$. Then H_κ is $\dot{\mathbb{Q}}^{H_{<\kappa}}$-generic over $V[G_{<\kappa}]$. Let $G = H_{<\kappa}$, $\mathbb{P} = \dot{\mathbb{Q}}^{H_{<\kappa}}$, $\mathcal{D} = (D_\alpha \mid \alpha < \omega_1) = \dot{\mathcal{D}}^G$.

Now consider $j(\mathbb{P}_\kappa)$. $\mathbb{P}_\kappa$ is an initial segment of $j(\mathbb{P}_\kappa)$ which in turn is an iteration of length $j(\kappa)$ in N. Hence we can find some $q \leq p^\frown \mathbb{1}_{j(\kappa)}$ that chooses $\mathbb{Q}$ from the lottery sum in the κ-th step. As in the proof of Theorem 1.3.10, j lifts to an embedding $j^* : M[G] \to N^* = N[H]$ by mapping $j^*(\sigma^G) = j(\sigma)^{G*H*I}$.

Since $j^*, H \in N^*$, the set $j^*[H]$ is an element of N^* and is directed, hence it generates a filter on $j^*(\mathbb{Q})$. Since H is $\mathbb{P}$-generic over M, for each $D \in \mathcal{D}$, there is some $x_D \in D \cap H$. Hence, by elementarity, $N^* \models j^*(x_D) \in j^*(D)$. Thus the filter generated by $j^*[H]$ is $(j^*(\mathbb{Q}), j^*(\mathcal{D}))$-generic. Again by elementarity, there must be a $(\mathbb{Q}, \mathcal{D})$-generic filter in $M[G]$. This filter would also be in $V[G]$ and contradict that $\mathbb{Q}$ is a counterexample to AAFA($\mathfrak{c}$). □

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